ADVANCED SUPERCONDUCTING MAGNETS INVESTIGATION

First Quarterly Progress Report

AVCO EVERETT RESEARCH LABORATORY
a division of
AVCO CORPORATION
Everett, Massachusetts

10 November 1966 through 10 February 1967

Contract NAS 8-21037

prepared for

GEORGE C. MARSHALL SPACE FLIGHT CENTER
NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
Huntsville, Alabama

I. INTRODUCTION

Considerable theoretical effort has been expended in the last quarter in refining and extending the work on stabilized conductors and multi-dimensional effects reported in Hoag, et al. [1]. This is in line with our goal of striving to attain a better understanding of the operation of high field superconducting coils and of predicting coil performance accurately from information regarding short sample terminal characteristics. All analyses concern the behavior of a short sample of superconductor which has been stabilized by placing it in electrical and thermal contact with a normally conducting substrate. This composite conductor is immersed in a liquid helium bath and exposed to an externally applied magnetic field. It will be assumed throughout that the current-carrying capacity of the superconductor is given by

$$\frac{I_s}{I_c} = 1 - \frac{T - T_b}{T_c - T_b} \tag{1}$$

where:

I_s = current carried by the superconductor

T = temperature of the superconductor

I_c = critical current corresponding to the bath temperature, T_b and the externally applied magnetic field

T_c = critical temperature corresponding to zero current and the externally applied field

To classify the analyses in this area and to clarify some of the inherent assumptions, it is advantageous to begin by writing the steady-state energy equation for a material element.

$$0 = W_i + \nabla \cdot (k\nabla T) - \frac{h}{P'}(T-T_b) \tag{2}$$

where:

W_i = heat generated per unit volume

T = temperature of the element

k = thermal conductivity (in general, a function of T)

h = heat transfer coefficient between the surface of the element and its environment; in general a function of temperature

T_b = temperature of the environment

∇ = vector differential operator "del"

P' = (volume of the element)/(heat transfer surface area)

This is the fundamental governing equation in each of the cases to be discussed. Each model is chosen so as to exemplify a single effect or refine an earlier analysis.

II. ANALYSES INVOLVING "ZERO" DIMENSIONS

The term zero dimensions will hereafter refer to the neglect of the second term in Eq. (2) and corresponds to the assumption that there is no spatial variation of temperature. Such an analysis was presented in detail in Hoag, et al. [2] for the case where k is constant, W_i is the sum of the ohmic dissipation and an additional specified heat source per unit volume, and the substrate is assumed to be in good electrical and thermal contact with the superconductor. The following section presents a simplified method in which the variation of heat transfer coefficient with temperature is considered.

A. Variable Heat Transfer Coefficient

Since the heat transfer from a conductor to liquid helium is temperature dependent, especially if the surface temperatures are high enough so that transition from nucleate to film boiling occurs, it is important to evaluate these effects and to determine whether (and how) the concepts developed using constant h analyses must be modified.

In the current sharing situation, the voltage per unit length developed in a conductor composed of a superconductor and a substrate in good thermal and electrical contact is given by

$$v = \frac{\rho}{A} I \; f \qquad (3)$$

where:

$\frac{\rho}{A}$ = resistance per unit length of substrate

I = total conductor current

f = fraction of the total current flowing in the substrate

Using Eq. (1), the fractional current in the normal conductor is

$$f = 1 - \frac{I_s}{I} = 1 - \frac{I_c}{I} \left(1 - \frac{T - T_b}{T_c - T_b}\right) \tag{4}$$

If the conductor has a cooled perimeter, P, then the heat flux per unit area is

$$q'' = \frac{vI}{P} = \frac{\rho I^2 f}{PA} \tag{5}$$

Substituting Eq. (4) in Eq. (5) yields

$$q'' = \frac{\rho I_c^2}{PA} \left[1 - \frac{I_c}{I} \left(1 - \frac{T - T_b}{T_c - T_b}\right)\right] \left(\frac{I}{I_c}\right)^2 \tag{6}$$

If we are given $\frac{\rho}{A}$, T_c, and I_c *, together with an empirical curve of q'' versus surface temperature rise, then a family of curves with I/I_c as parameter may be plotted on the same graph by using Eq. (6). The intersections of these curves with the original empirical curve represent valid operating points and, since the heat flux at each value of I/I_c is thus determined, the voltage-current terminal characteristics can be found from

$$v = \frac{q'' P}{I_c} \left(\frac{I_c}{I}\right) \tag{7}$$

Note that the q'' corresponding to a given value of I/I_c is determined from one of the points of intersection on the graph.

The characteristic form of the voltage current plot then indicates whether the particular coil is stable.

B. Effect of Thermal Contact Resistance

In all previous analyses, it has been stated that the superconductor is in good thermal contact with the substrate. This is equivalent to assuming that there is no discontinuity in temperature at the interface. This section presents a simple model which exhibits the effects of a finite thermal contact resistance.

*In a short sample test involving a constant externally applied field, $\frac{\rho}{A}$, T_c and I_c correspond to the values at the applied field. In a superconducting magnet, these variables are functions of the current, I, since this determines the field strength.

Using the notation of the previous sections, the heat generated per unit length in the superconducting portion of a composite conductor is

$$q_s = v\,I\,(1-f) = \frac{\rho I^2}{A} f\ (1-f) \tag{8}$$

and the heat generated per unit length in the substrate is

$$q_n = v\,I\,f = \frac{\rho}{A} I^2\ f^2 \tag{9}$$

Assuming that the heat generated in the superconductor must be transferred to the normal substrate through a contact heat transfer coefficient, h_i, over an interface perimeter, P_i, the difference in temperature between the superconductor and substrate is

$$T_s - T_n = \frac{\rho I^2}{h_i P_i A}\ f\,(1-f) \tag{10}$$

The total heat generated in the conductor must be transferred to the liquid helium bath over a perimeter, P, and with a heat transfer coefficient, h, which is assumed to include the thermal resistance of any insulation. The temperature rise of the substrate above the liquid helium bath is, then:

$$T_n - T_b = \frac{\rho I^2 f}{h\,P A} \tag{11}$$

Adding Eqs. (10) and (11) yields the temperature difference between the superconductor and the bath.

$$T_s - T_b = \frac{\rho I^2}{h\,P\,A}\ f + \frac{\rho I^2}{h_i P_i A}\ f\ (1-f) \tag{12}$$

The following equation may now be developed using Eqs. (1) and (12) and the definition of f

$$\alpha_i \left(\frac{I}{I_c}\right)^2 f^2 - \left[(\alpha + \alpha_i)\ \frac{I}{I_c} - 1\right]\left(\frac{I}{I_c}\right) f + 1 - \frac{I}{I_c} = 0 \tag{13}$$

where:

$$\alpha = \frac{\rho\, I_c^2}{h\,P\,A\,(T_c - T_b)}$$

$$a_i = \frac{\rho I_c^2}{h_i P_i A (T_c - T_b)}$$

Equation (13) may be solved for f. Then with Eq. (1) and the expression for the voltage, the voltage per unit length and the superconductor temperature may be determined. The results are:

$$f = \frac{1}{2}\left\{\frac{a + a_i}{a_i} - \frac{1}{a_i \cdot \tau} \pm \sqrt{\left(\frac{a + a_i}{a_i} - \frac{1}{a_i \tau}\right)^2 - 4\left(\frac{1 - \tau}{a_i \tau^2}\right)}\right\} \tag{14}$$

$$V = \frac{vA}{\rho I_c} = f\,\tau = \frac{1}{2}\,\tau\left\{\frac{a + a_i}{a_i} - \frac{1}{a_i \tau} \pm \sqrt{\left(\frac{a + a_i}{a_i} - \frac{1}{a_i \tau}\right)^2 - 4\left(\frac{1-\tau}{a_i \tau^2}\right)}\right\} \tag{15}$$

$$\theta = \frac{T_s - T_b}{T_c - T_b} = 1 - \tau + \tfrac{1}{2}\,\tau\left\{\frac{a + a_i}{a_i} - \frac{1}{a_i \cdot \tau} \pm \sqrt{\left(\frac{a + a_i}{a_i} - \frac{1}{a_i \tau}\right)^2 - 4\left(\frac{1 - \tau}{a_i \tau^2}\right)}\right\} \tag{16}$$

where:

$$\tau = \frac{I}{I_c}$$

In searching for a stability criterion, we require that condition which corresponds to the onset of a positive resistance in a composite conductor when the current reaches the short sample critical current of the superconductor. We, therefore, consider $\frac{\partial V}{\partial \tau}$ at $\tau = 1$ and $f = 0$. This may be shown to be

$$\left.\frac{\partial V}{\partial \tau}\right|_{\substack{\tau = 1 \\ f = 0}} = \frac{1}{1 - (a + a_i)} \tag{17}$$

In order for the slope of voltage vs current to be positive at the critical current, then, it is necessary that

$$a + a_i < 1 \tag{18}$$

Equation (18) is the required condition for stability.

Typical behavior is illustrated in Fig. 1 which was obtained by plotting Eq. (15) for $\alpha = 0.5$ and various values of α_i. For this case, if $\alpha_i \leq 0.5$ then the voltage will remain zero until τ reaches one. At this point, a voltage will appear and will increase continuously and controllably as τ increases further. For $\alpha_i > .5$ (in general, for $\alpha + \alpha_i > 1$), the voltage current curves are double valued. Since operation on the part of a particular curve (specified α, α_i) which corresponds to negative resistance is impossible, it may be expected that for

1) $\tau \leq \tau_R = \tau_R(\alpha, \alpha_i)$, all current will flow in the superconductor,

2) for $\tau_R \leq \tau < 1$, however, the current will transfer to the substrate if a disturbance occurs. The voltage will then rise discontinuously from zero to that determined by the positive resistance portion of the curve specified by α and α_i. In order to recover to a fully superconducting state, it is then necessary to lower τ to τ_R where the voltage will discontinuously decrease to zero. (Note that if no disturbance occurs, then the voltage must still rise discontinuously when τ reaches one).

The function $\tau_R = \tau_R(\alpha, \alpha_i)$ may be found from Eqs. (14), (15), or (16) under the condition that f, V, or θ must be single valued at $\tau = \tau_R$. These "recovery" conditions may be shown to be:

$$\tau_R = \left(\frac{I}{I_c}\right)_R = \frac{(\alpha - \alpha_i) \pm 2\sqrt{-\alpha\,\alpha_i + \alpha_i(\alpha + \alpha_i)^2}}{(\alpha + \alpha_i)^2} \tag{19}$$

$$f_R = \sqrt{\frac{1 - \tau_R}{\alpha_i\,\tau_R^2}} \tag{20}$$

$$V_R = \frac{v_R A}{\rho\, I_c} = f_R\,\tau_R \tag{21}$$

$$\theta_R = \left.\frac{T_s - T_b}{T_c - T_b}\right|_R = \left[\alpha - \alpha_i(1 - f_R)\right] f_R\,\tau_R^2 \tag{22}$$

where it is necessary that $0 \leq \tau_R \leq 1$.

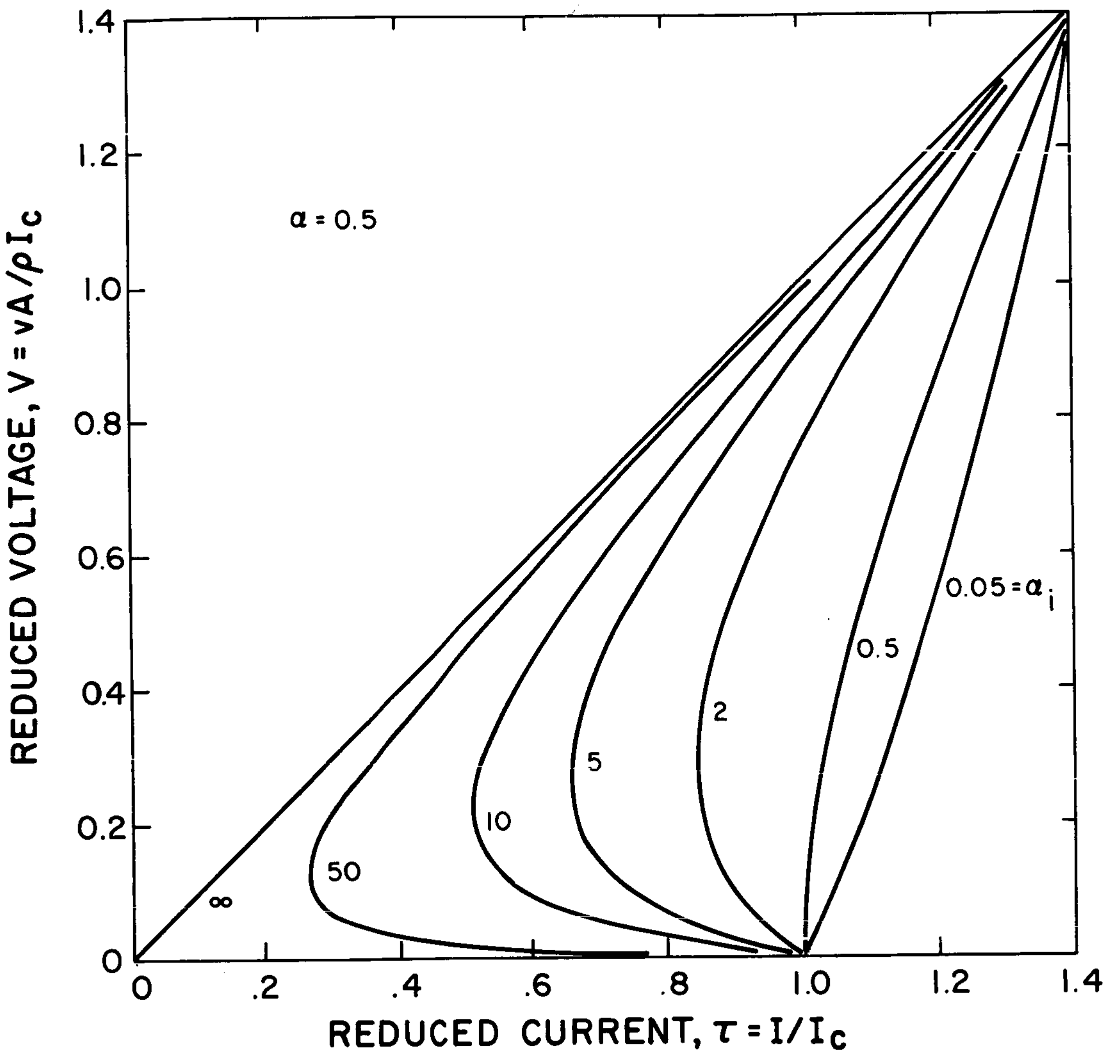

Fig. 1 Illustration of the Effect of Thermal Contact Resistance Between the Superconductor and Substrate.

The results of the above analyses are best summarized graphically in maps (Figs. 2 and 3) having α and α_i as coordinates and lines of constant recovery conditions on it, so that given α and α_i, then the recovery current, its distribution, and conductor voltage per unit length at recovery can be read off directly. Conversely, the graph can be used to determine the effective values of α and α_i from the fraction of short sample current at recovery and the voltage per unit length just prior to recovery.

It is important to determine the effect on the stability criterion due to the possibility of self-generated heat as may occur, for example, in a contact. From another viewpoint, a heater may be used for diagnostic purposes. To consider this effect, assume that any heat generated flows into the substrate and from there to the helium bath, so that the temperature difference between the superconductor and substrate is still given by Eq. (10).

If an amount of heat, q_{ho}, is generated per unit length, then Eq. (12) becomes

$$T_s - T_b = \frac{\rho I^2 f}{h P A} + \frac{\rho I^2}{h_i P_i A} f(1-f) + \frac{q_{ho}}{hP} \tag{23}$$

A relation for f, τ, α, and α_i analogous to Eq. (13) may now be found.

$$\alpha_i \tau^2 f^2 - \left[(\alpha + \alpha_i)\tau - 1\right] \tau f + 1 - \tau - Q_{ho} = 0 \tag{24}$$

where

$$Q_{ho} = \frac{q_{ho}}{h P (T_c - T_b)}$$

By setting f = 0 in Eq. (24), it is clear that current begins to transfer into the substrate when

$$\tau = 1 - Q_{ho} \tag{25}$$

If we then require that $\frac{\partial V}{\partial \tau}$ at $\tau = 1 - Q_{ho}$ and f = 0 represent positive resistance, the stability criterion will be established. In this way, it may be shown that

$$\left.\frac{\partial V}{\partial \tau}\right|_{\substack{\tau = 1 - Q_{ho} \\ f = 0}} = \frac{1}{1 - (1 - Q_{ho})(\alpha + \alpha_i)} \tag{26}$$

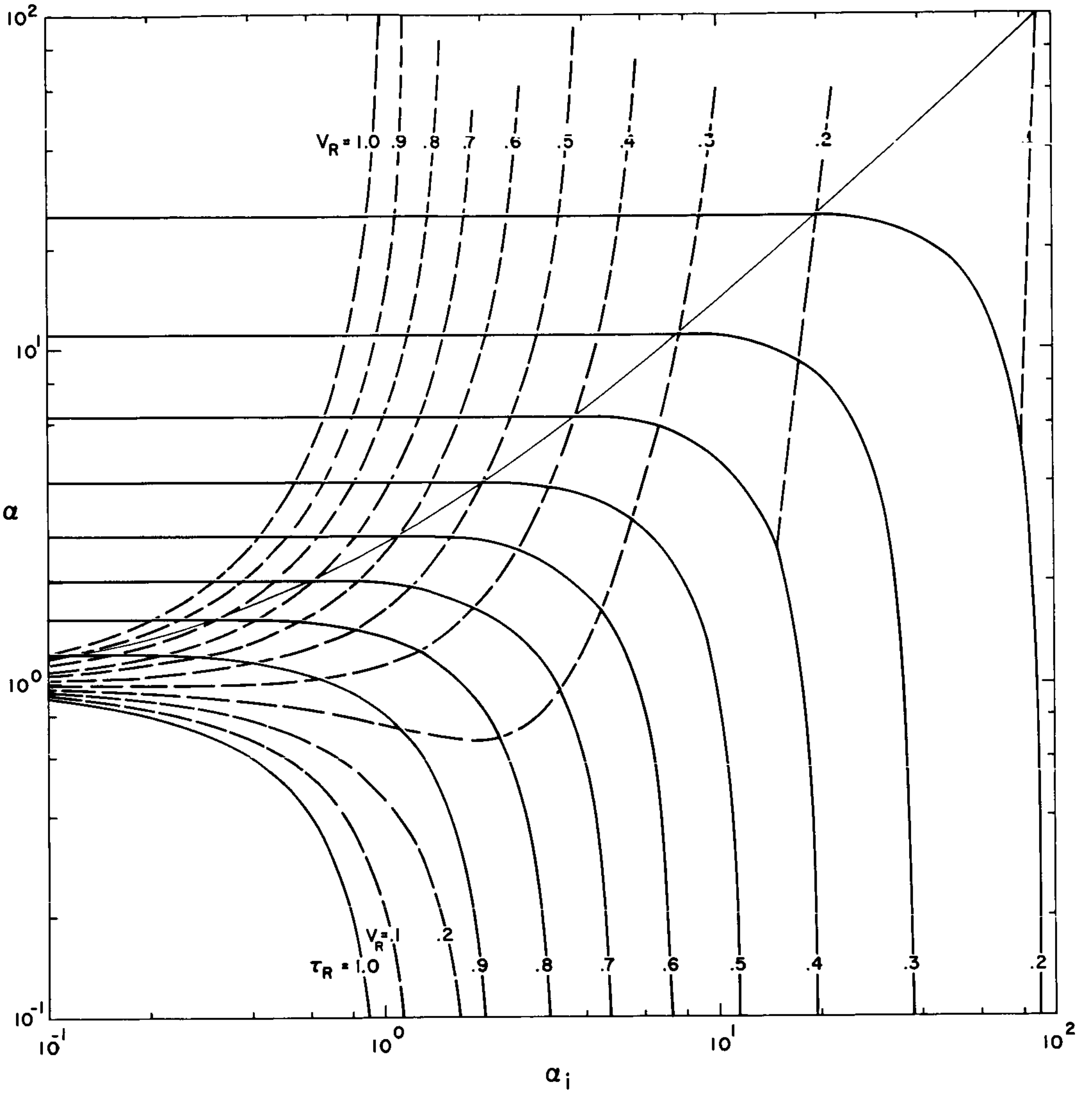

Fig. 2 Recovery Current τ_R and Recovery Voltage V_R as a Function of α and α_i.

A7459

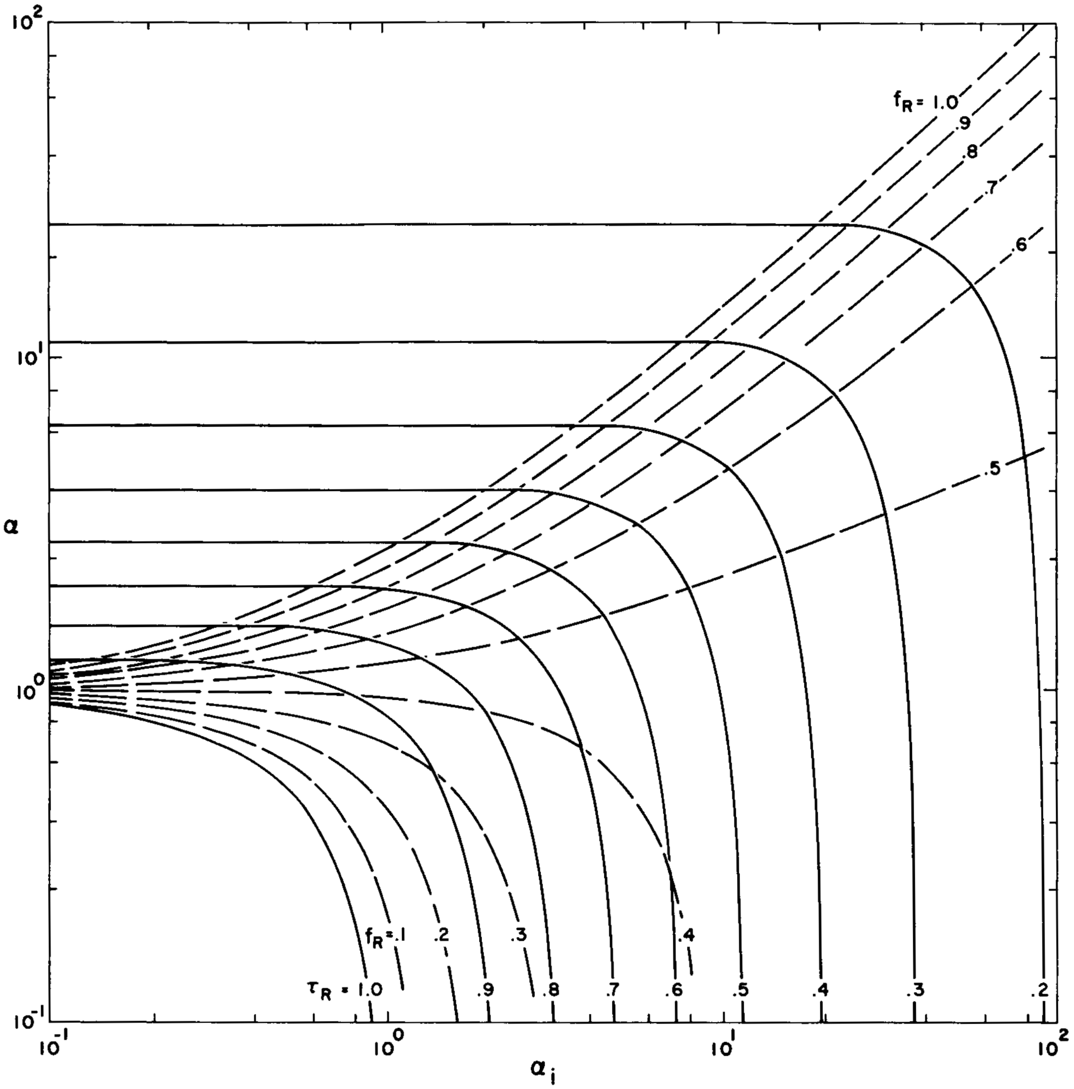

Fig. 3 Recovery Current and Recovery Current Distribution as a Function of α and α_i.

A7460

Hence, stability requires that

$$(1 - Q_{ho})\ (\alpha + \alpha_i) = \tau\ (\alpha + \alpha_i) < 1 \tag{27}$$

This implies that a conductor, which is subjected to enough heating to produce a normal region, will exhibit stable, positive resistance behavior for current $\tau < \frac{1}{\alpha + \alpha_i}$.

III. ONE-DIMENSIONAL EFFECTS

Initial results regarding the behavior of a one-dimensional composite conductor consisting of a superconductor in good electrical and thermal contact with a normal conductor were reported in Hoag, et al. [3]*. In that report, the governing equation (Eq. (2))was written as

$$0 = \frac{\rho f I^2}{A} + kA\ \frac{\partial^2 T}{\partial x^2} - h\,P\,(T - T_b) \tag{28}$$

where h and k are assumed constant, P is the cooled perimeter, A is the cross-sectional area, and f is the fraction of the total current I which flows in the substrate. The latter is characterized by the resistivity ρ.

Equation (28) is then made dimensionless in terms of

$$\theta = \frac{T - T_b}{T_c - T_b}$$

$$\zeta = \frac{x}{x_o}\ , \quad x_o = \sqrt{\frac{kA}{hP}}$$

$$\tau = \frac{I}{I_c}$$

$$\alpha = \frac{\rho I_c^2}{hPA\,(T_c - T_b)}$$

*In Hoag, this was referred to as a two-dimensional analysis. We will henceforth adopt the convention that an analysis is one-dimensional if the form of the ∇^2 operator in Eq. (2) is $\frac{\partial^2}{\partial \zeta^2}$, two-dimensional if it is $\frac{1}{\zeta}\ \frac{\partial}{\partial \zeta}(\zeta\,\frac{\partial}{\partial \zeta})$, and three dimensional if it is $\frac{1}{\zeta^2}\ \frac{\partial}{\partial \zeta}\ (\zeta^2\ \frac{\partial}{\partial \zeta})$.

The conductor is subjected to a disturbance in the form of a specified heat input q_{h1} at the origin. In dimensionless form, this is written:

$$Q_{h1} = \frac{q_{h1}}{2\sqrt{hPkA}\,(T_c - T_b)}$$

The form of the governing equation is dependent on f and solutions for the temperature are found for two cases:

1) The two-region case, where Q_{h1} at the orgin is such that $0 < f < 1.0$ at the origin, hence, the conductor operates in a condition such that the current is shared between the superconductor and the substrate for $\zeta < \Delta\zeta$ and all the current is in the superconductor for $\zeta > \Delta\zeta$.

2) The three-region case, where Q_{h1} is high enough so that $f > 1.0$ at $\zeta = 0$, hence, the conductor operates in a condition such that essentially all the current is in the substrate for $0 < \zeta < \zeta_1$, the current is shared for $\zeta_1 < \zeta < \zeta_1 + \Delta\zeta$, and all the current flows in the superconductor for $\zeta > \zeta_1 + \Delta\zeta$.

Expressions for the voltage developed across the "terminals" of the conductor are determined next and three classes of behavior are then shown for a particular case by plotting the voltage vs heat input at the origin for various values of the parameter α and for constant current. These curves are shown in Fig. 4 for later reference.

Since that time, a detailed analysis of the stability of the above configuration has been carried out and stability criterion have been developed analytically. This consisted of determining the conditions under which $\frac{\partial V}{\partial Q_{h1}} \rightarrow \infty$. For the particular case of $\tau = 0.5$ shown in Fig. 4, it may be seen that $\frac{\partial V}{\partial Q_{h1}}$ is always finite when α is small enough. This represents single valued, stable operation in which the voltage is zero until Q_{h1} reaches a certain value, then it increases monotonically with positive slope. For α large, there is a maximum value of Q_{h1} for which a solution exists that is, increasing Q_{h1} at constant current, results in an uncontrolled quench when Q_{h1} reaches this maximum value. Between these two extremes is a range of α in which hysteretic behavior is observed. For these cases, operation is exemplified in Fig. 4 by the curve g-a-f-b-c-d-c-e-f-a-g as Q_{h1} is increased from zero to that value corresponding to point d and back to zero.

In general, the type of behavior is dependent on the value of α, τ, and Q_{h1}. It may be shown that $\frac{\partial V}{\partial Q_{h1}}$ must remain finite for $\alpha\tau^2 < 1$, hence, as indicated in Fig. 5, the region bounded by the axes and

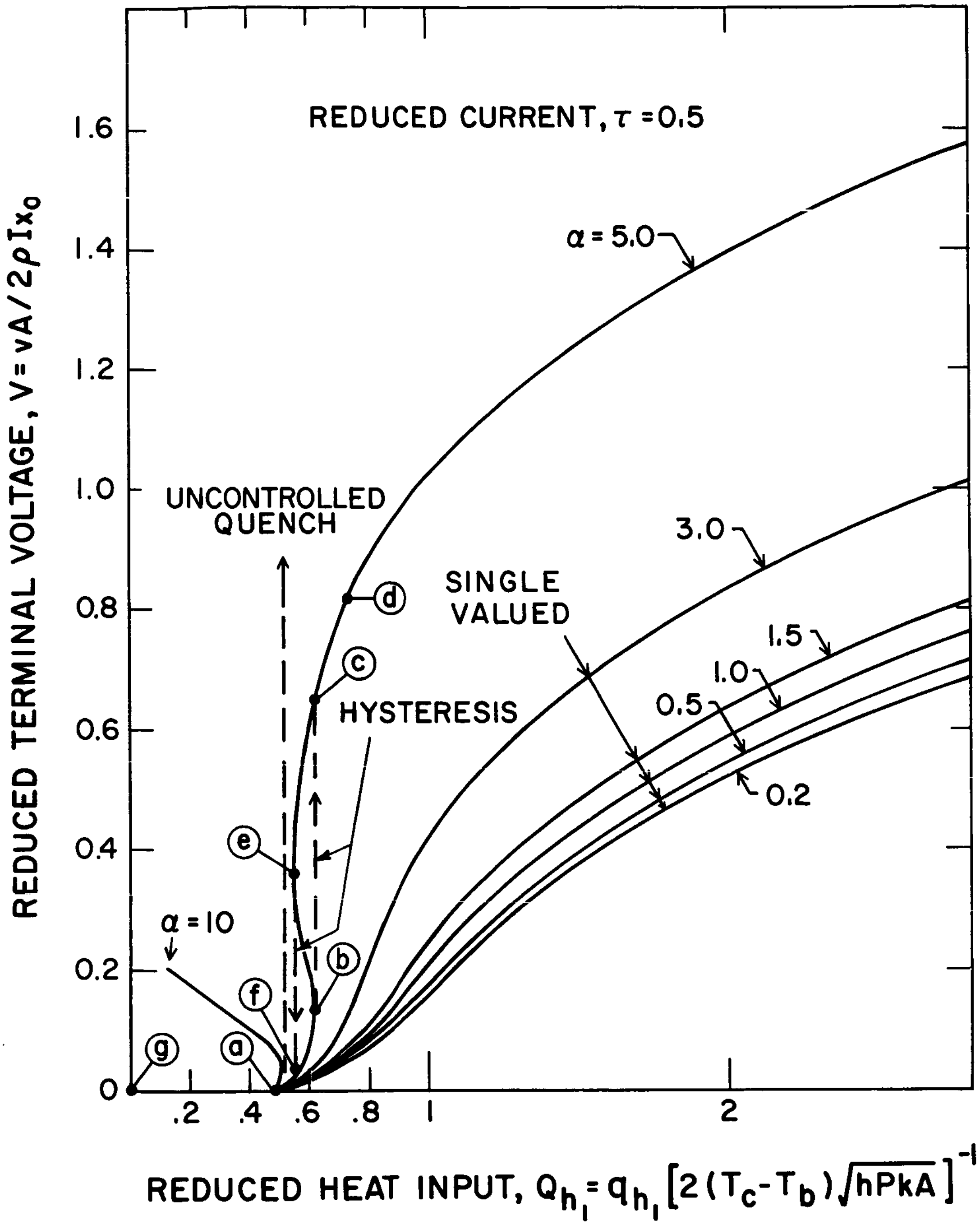

Fig. 4 Terminal Voltage as a Function of Heat Input. Three classes of behavior are shown.

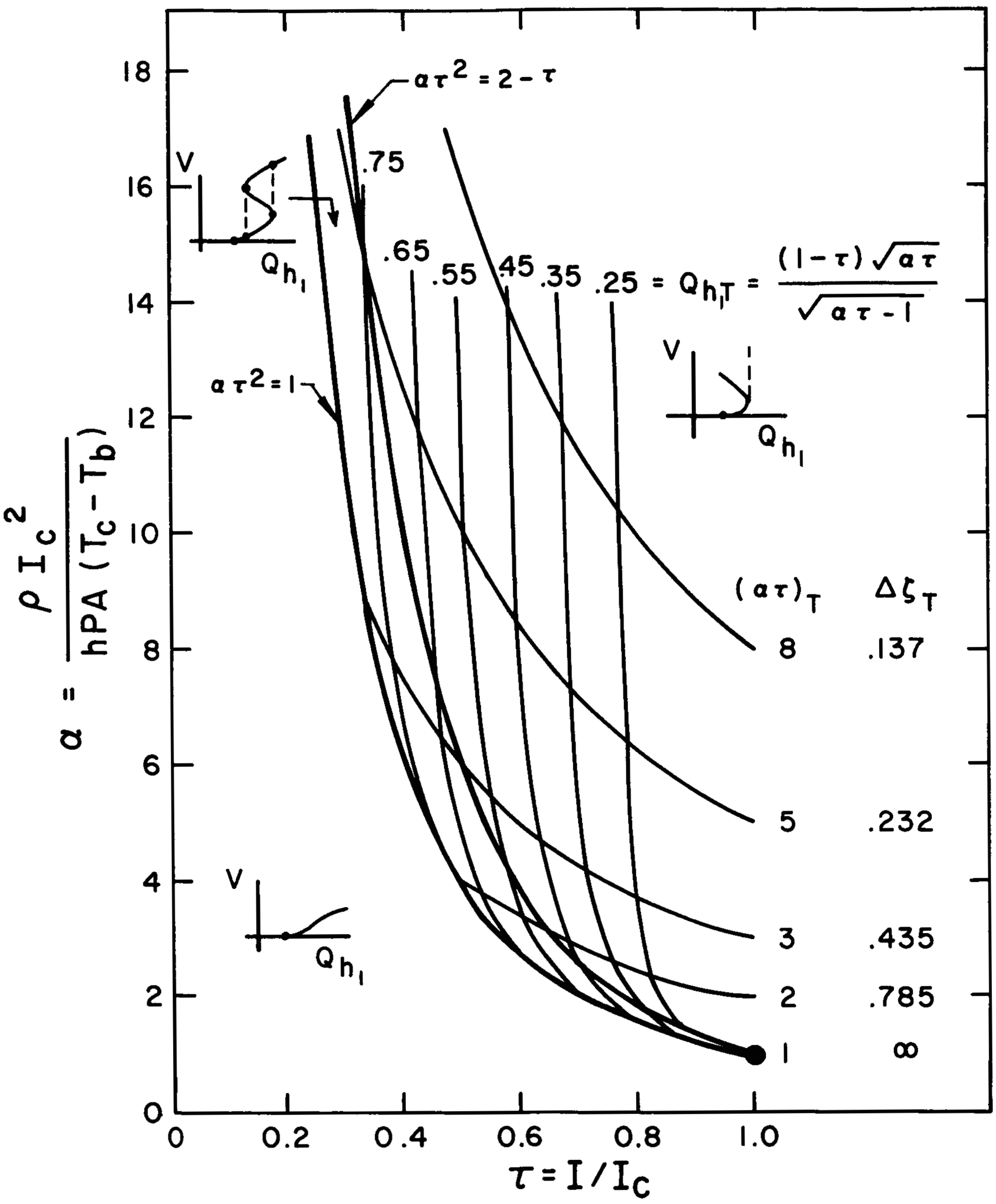

Fig. 5 Stability Diagram Indicating Range of α, τ over which Stable, Hysteretic, or Unstable Behavior Occurs.

A7451

the curve $\alpha \tau^2 = 1$ represents stable operation. For $\alpha\tau^2 > 1$, $\frac{\partial V}{\partial Q_{hl}} \rightarrow \infty$ when Q_{hl} reaches a "take-off" value given by:

$$Q_{hlT} = \frac{\sqrt{\alpha \tau}\ (1 - \tau)}{\sqrt{\alpha \tau - 1}} \tag{29}$$

This corresponds to attaining a "take-off" length

$$\Delta\zeta_T = (\alpha \tau - 1)^{-\frac{1}{2}} \arctan \left[(\alpha\tau - 1)^{-\frac{1}{2}}\right] \tag{30}$$

In addition, it may be shown that $\zeta_o \rightarrow \infty$ when $\alpha \tau^2 = (2 - \tau)$. This is the quench condition, that is $V \rightarrow \infty$. The curve $\alpha\tau^2 = 2 - \tau$ as well as curves for constant Q_{hlT} and $\Delta\zeta_T$ are also plotted in Fig. 5. This is interpreted as follows:

1) For α, τ such that $\alpha\tau^2 < 1$, the behavior of the system is stable in the sense that $\frac{\partial V}{\partial Q_{hl}}$ is always finite and positive.

2) For α, τ such that $\alpha \tau^2 > (2 - \tau)$, the behavior is unstable since the voltage will rise as Q_{hl} is increased until $Q_{hl} = Q_{hlT}$ (or, correspondingly, $\Delta\zeta = \Delta\zeta_T$), at which time $V \rightarrow \infty$.

3) For α, τ such that $1 < \alpha\tau^2 < (2 - \tau)$, the behavior is hysteretic in that the voltage continuously rises to a value at which $Q_{hl} = Q_{hlT}$ when it increases discontinuously to another finite value. The conductor will then not regain its superconducting state until Q_{hl} is lowered through a recovery value ($Q_{hlR} < Q_{hlT}$) where the voltage decreases discontinuously to zero or a relatively small value.

Figure 6 is a graph of Q_{hl} vs τ for $\alpha = 3$ and clearly represents the different regions of operation for different values of τ.

1) For $\tau < \frac{1}{\sqrt{\alpha}}$, the stable range, no voltage is exhibited until Q_{hl} reaches the value corresponding to the intersection of the τ = constant, line of operation with the $Q_{hl} = (1 - \tau)$ curve which represents the onset of resistance. If Q_{hl} continues to increase, the voltage increases in a stable manner as shown in the insert.

2) If the line of operation lies in the unstable range, there is no voltage until $Q_{hl} = (1 - \tau)$, then the voltage increases controllably until the intersection with the Q_{hlT} curve at which time quench occurs (i.e., $v \rightarrow \infty$).

3) The hysteretic range may be subdivided into two-regions.

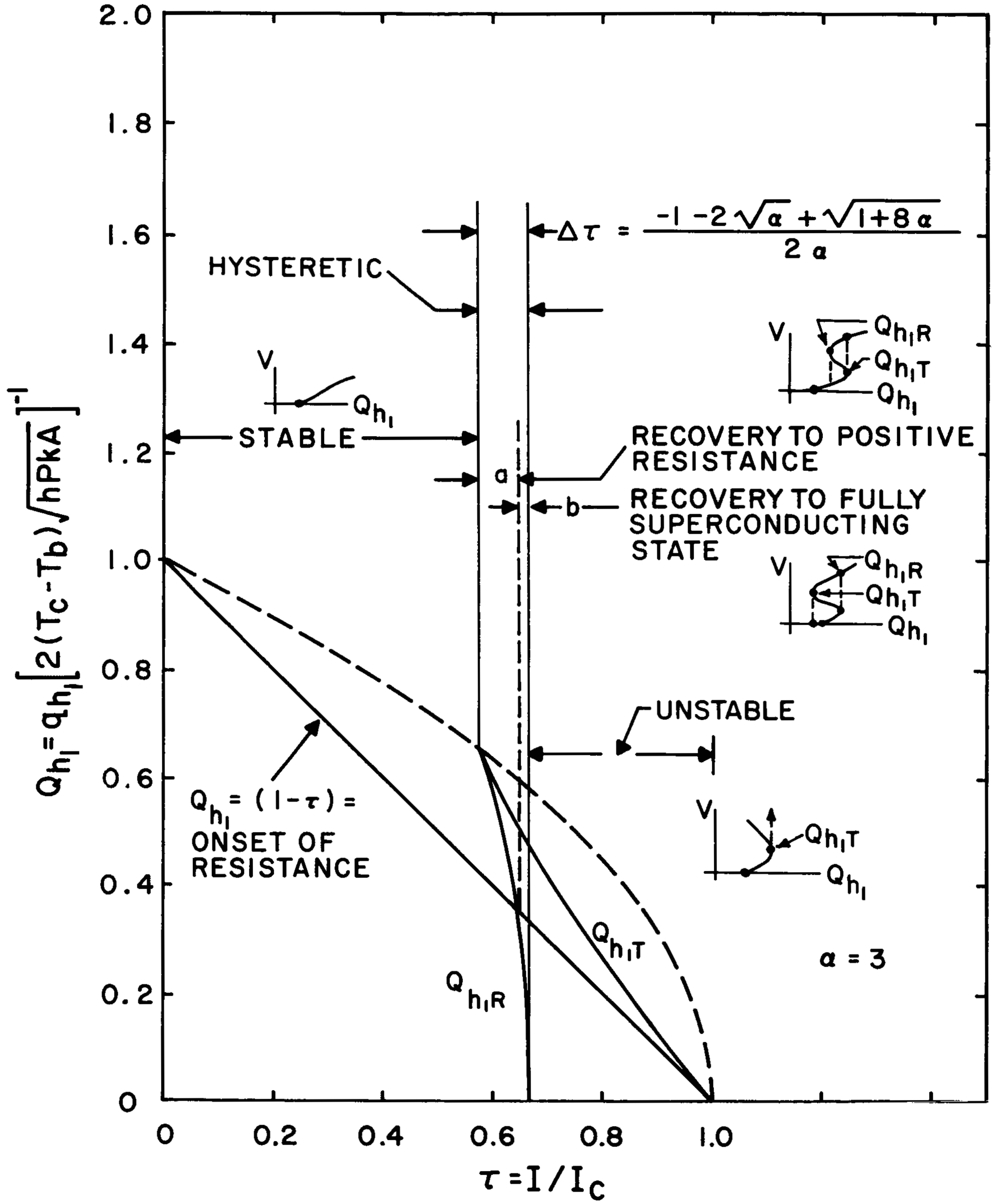

Fig. 6 Heat input vs Current: an indicator of the type of behavior for different τ as α is held fixed.

A7450

3a) For τ in region "a", there is again no voltage until Q_{h1} = $(1 - \tau)$. As Q_{h1} is increased, V increases until $Q_{h1} = Q_{h1T}$ when a discontinuous rise in voltage occurs. If Q_{h1} is then reduced to the value indicated by the intersection with the Q_{h1R} curve, V will decrease discontinuously and a recovery will be made to a condition of positive resistance.

3b) Operation in this range is similar to that in region "a" with the important difference that recovery is made to a condition of zero voltage, that is, to the fully superconducting state. The relation between α and τ for the condition where Q_{h1R} is equal to $(1 - \tau)$ may be shown to be

$$\left(\frac{\alpha\tau^2 - 1}{1 - \tau}\right)^2 \alpha\tau = 1 \tag{31}$$

A plot of the values of the critical voltages for the case of $\alpha = 5$ is shown in Fig. 7.

1) For τ in the region indicated, operation is stable.

2) For τ in the unstable region, the voltage may increase as far as the curve labeled V_T at which quench will occur.

3a) In the hysteretic "a" region, the voltage may be increased to V_T where it will discontinuously rise to the value indicated by the V' curve. If Q_{h1} is then decreased, the voltage will decrease to the value indicated by the V_R curve. At this point, it will decrease discontinuously to the value represented by the intersection of the line of operation with the V'' curve. Further decrease in Q_{h1} leads to a continuous decrease in V to zero.

3b) In the hysteretic "b" region, operation is analogous to that in the "a" region except that the voltage drops discontinuously to zero at recovery.

Figure 8 is similar to Fig. 7 in that it shows the take-off and recovery voltages. This is done for a range of values of α. Several corresponding points are labeled in the two plots to aid interpretation of Fig. 8.

IV. THREE-DIMENSIONAL EFFECTS

It may be expected that the effects exhibited by a multi-dimensional treatment of the stabilization problem would be qualitatively similar to those which are indicated by the zero and one-dimensional analyses, however, when attempting to obtain experimental correlation, it is well to have an indication of any changes in the stability criterion which may occur as a result of the actual three-dimensional nature of the heat transfer problem.

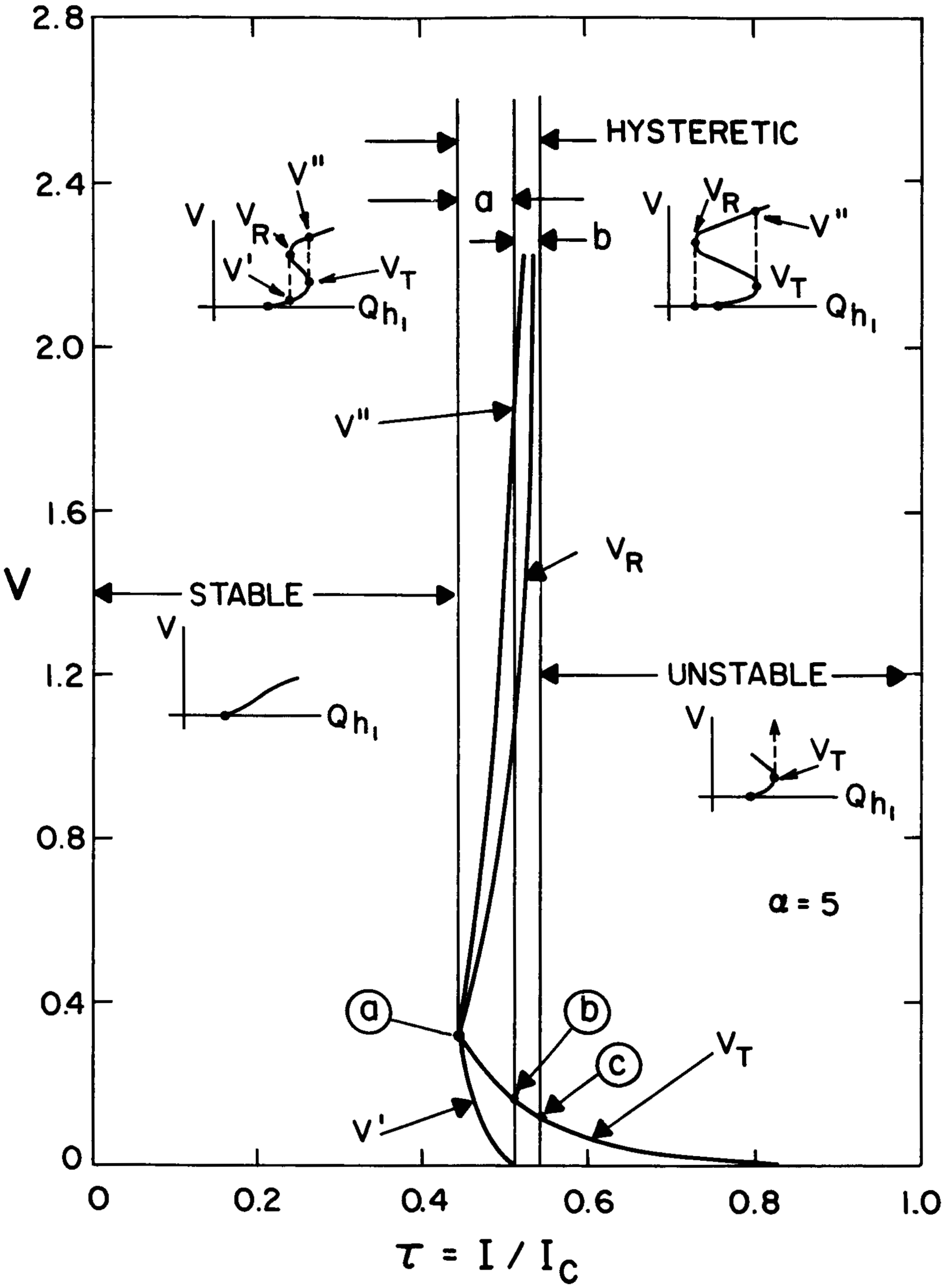

Fig. 7 Terminal Voltage vs Current: the behavior of the voltage for various τ and for α fixed.

A7427

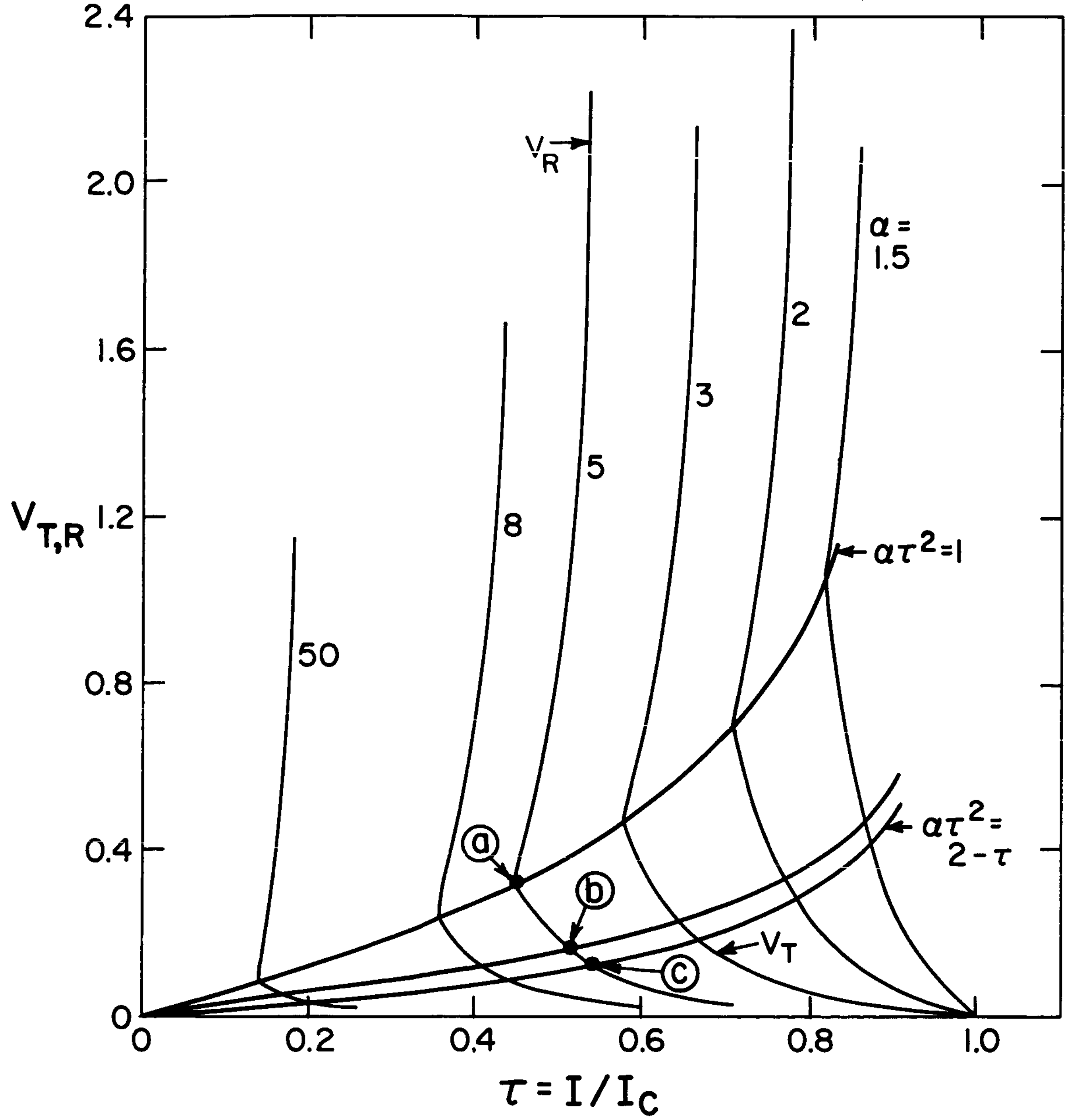

Fig. 8 Takeoff and Recovery Voltages as a Function of τ for Various α.

A7453

IVa. THE $\alpha \rightarrow \infty$ CASE

This section deals with the limiting case where there is no cooling provided by helium within the coil. The governing relation is Eq. (2) with the last term neglected. A point heat source is assumed at the origin. This injection of a finite amount of heat requires an infinite temperature gradient at the origin and leads to a condition where only a three-region solution is possible:

1) A region $0 \leq \zeta \leq \zeta_o$ in which all the current flows in the substrate,

2) A region $\zeta_o \leq \zeta \leq \zeta_1$ in which the current is shared, and

3) A region $\zeta > \zeta_1$ where all the current flows in the superconductor.

Begin by writing Eq. (2) in its three-dimensional form

$$0 = W_i + \frac{k}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r}\right) \tag{32}$$

We now specify the ohmic dissipation per unit volume to be

$$W_i = \lambda \frac{\rho f I^2}{A^2} \tag{33}$$

where

$\frac{\rho f I}{A}$ = voltage per unit length

$\lambda = \frac{\text{wire volume}}{\text{total volume}}$

λ thus allows for the packing factor or any volume not producing heat (e.g., insulation). If we now use Eq. (1) and

$$\theta = \frac{T - T_b}{T_c - T_b} \tag{34}$$

$$\tau = \frac{I}{I_c} \tag{35}$$

$$\zeta = \frac{r}{r_o} \tag{36}$$

$$r_o^2 = \frac{kA^2 (T_c - T_b)}{\lambda \rho I_c^2} \tag{37}$$

then the governing equation in region 1 where f = 1, may be shown to be

$$\frac{1}{\zeta^2} \frac{\partial}{\partial \zeta} (\zeta^2 \frac{\partial \theta_1}{\partial \zeta}) + \tau^2 = 0, \ 0 < \zeta < \zeta_o \tag{38}$$

In region 2 ($0 < f < 1$) the current is shared and the situation is governed by

$$\frac{1}{\zeta^2} \frac{\partial}{\partial \zeta} (\zeta^2 \frac{\partial \theta_2}{\partial \zeta}) + \tau \theta_2 - \tau (1 - \tau) = 0, \ \zeta_o < \zeta < \zeta_1 \tag{39}$$

The proper equation for region 3 ($f = 0$), in which all the current is in the superconductor, is

$$\frac{1}{\zeta^2} \frac{\partial}{\partial \zeta} (\zeta^2 \frac{\partial \theta_3}{\partial \zeta}) = 0, \ \zeta > \zeta_1 \tag{40}$$

The boundary conditions for this configuration are:

$$\theta_3 \to 0 \qquad \text{as} \qquad \zeta \to \infty \tag{41}$$

$$\theta_2 = \theta_3 = (1 - \tau) \quad \text{and} \quad \frac{d\theta_2}{d\zeta} = \frac{d\theta_3}{d\zeta} \quad \text{at} \quad \zeta = \zeta_1 \tag{42}$$

$$\theta_1 = \theta_2 = 1 \quad \text{and} \quad \frac{d\theta_1}{d\zeta} = \frac{d\theta_2}{d\zeta} \quad \text{at} \quad \zeta = \zeta_o \tag{43}$$

$$\zeta^2 \frac{d\theta_1}{d\zeta} \to -Q'_{h3} \equiv - \frac{q_{h3}}{4 \pi k r_o (T_c - T_b)} \quad \text{as} \quad \zeta \to 0 \tag{44}$$

It may be shown that the solution to these three differential equations subject to the given boundary conditions is

$$\theta_1 = \frac{\tau^2}{6} (\zeta_o^2 - \zeta^2) - Q'_{h3} (\frac{1}{\zeta_o} - \frac{1}{\zeta}) + 1 \tag{45}$$

$$\theta_2 = \zeta^{-1} \left[C \sin(\sqrt{\tau}\ \zeta) + D \cos(\sqrt{\tau}\ \zeta) \right] + (1 - \tau) \tag{46}$$

$$\theta_3 = (1 - \tau)\ \zeta_1\ \zeta^{-1} \tag{47}$$

where:

$$C = - \frac{(1 - \tau)}{\sqrt{\tau}} \cos(\sqrt{\tau}\ \zeta_1)$$

$$D = \frac{(1 - \tau)}{\sqrt{\tau}} \sin(\sqrt{\tau}\ \zeta_1)$$

The lengths ζ_1 and ζ_o are determined by

$$\sin \left[\sqrt{\tau} (\zeta_1 - \zeta_o) \right] = \frac{\zeta_o\ \tau^{3/2}}{1 - \tau} \tag{48}$$

and

$$\cos \left[\sqrt{\tau} (\zeta_1 - \zeta_o) \right] = - (1 - \tau)^{-1} \left[\tau - \frac{(\zeta_o \tau)^2}{3} - \frac{Q'_{h3}}{\zeta_o} \right] \tag{49}$$

Equations (48) and (49) may be manipulated to show that

$$Q'_{h3} = \zeta_o \left[\tau - \frac{(\zeta_o \tau)^2}{3} \pm \sqrt{(1 - \tau)^2 - \zeta_o^2\ \tau^3} \right] \tag{50}$$

The condition for which $\frac{\partial V}{\partial Q'_{h3}} \rightarrow \infty$ is of particular interest since it represents instability. V is closely related to ζ_o, however, so that it is only necessary to determine the "take-off" heat Q'_{h3T} or "take-off", length, ζ_{oT}, for which $\frac{\partial \zeta_o}{\partial Q'_{h3}} \rightarrow \infty$. By equating $\partial Q'_{h3} / \partial \zeta_o$ to zero, ζ_{oT} may be found to be given by:

$$\zeta_{oT}^6 + (\tau^2 + 2\tau - 1)\ \tau^{-3}\ \zeta_{oT}^4 - (\tau^2 - 4\tau + 2)\ \tau^{-4}\ \zeta_{oT}^2 + (1-\tau)^2 (1-2\tau)\tau^{-7} = 0 \tag{51}$$

Equation (51) may now be used together with Eq. (50) to find $Q'_{h3T} = Q'_{h3T}(\tau)$, which is plotted in Fig. 9. A process of curve fitting may then be used to show that the curve is closely approximated by

$$Q'_{h3T} \approx (0.72) \frac{(1 - \tau)^{0.88}}{\tau^{0.97}} \tag{52}$$

in the range of interest. For constant current operation, Q'_{h3} may be increased up to the value of Q'_{h3T} indicated by the intersection of the line of operation with the Q'_{h3T} curve. At that time $\zeta_o \rightarrow \infty$ and, in turn, $V \rightarrow \infty$. This behavior is illustrated in Fig. 10 which is a plot of Eq. (50) for selected values of τ. Operation on the dashed portion of the curves is not possible. Note that ζ_o remains finite as $\tau \rightarrow 0$. This results from the boundary condition at the origin which requires an infinite temperature gradient to "drive" a finite Q'_{h3}.

IVb. THE CASE OF ARBITRARY α

A study has been initiated concerning a configuration similar to that in the previous section with the important difference that all three terms are retained in Eq. (2). We, therefore, begin by writing

$$0 = W_i + \frac{k}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r}\right) - \frac{h}{P'} (T - T_b) \tag{53}$$

where P' is a characteristic length related to the cooling by the helium within the winding. It is essentially the ratio of the coil volume to the internal heat transfer surface area. Using Eqs. (1), (33), (34), (35), and the definition of f, together with

$$\zeta = \frac{r}{r_o}$$

$$\alpha = \frac{\lambda \rho I_c^2 P'}{h A^2 (T_c - T_b)} ,$$

and

$$r_o^2 = \frac{k x_o}{h} ,$$

it may be shown that the governing equation for:

1) The region where all the current is carried by the substrate (f = 1) is:

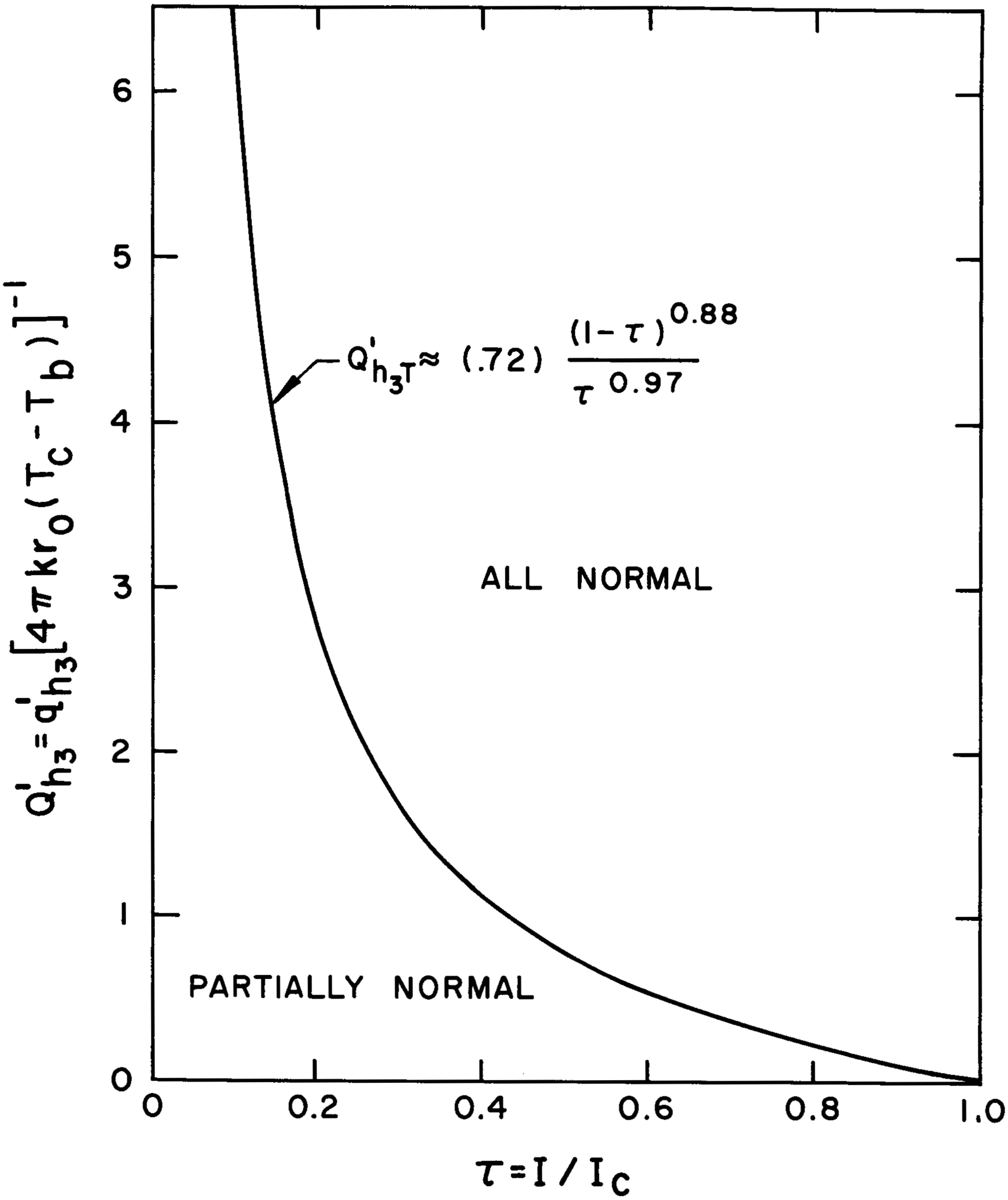

Fig. 9 Heat Input as a Function of Current; Q_{hT} is that value of Q_{nB} at which quench occurs.

A7452

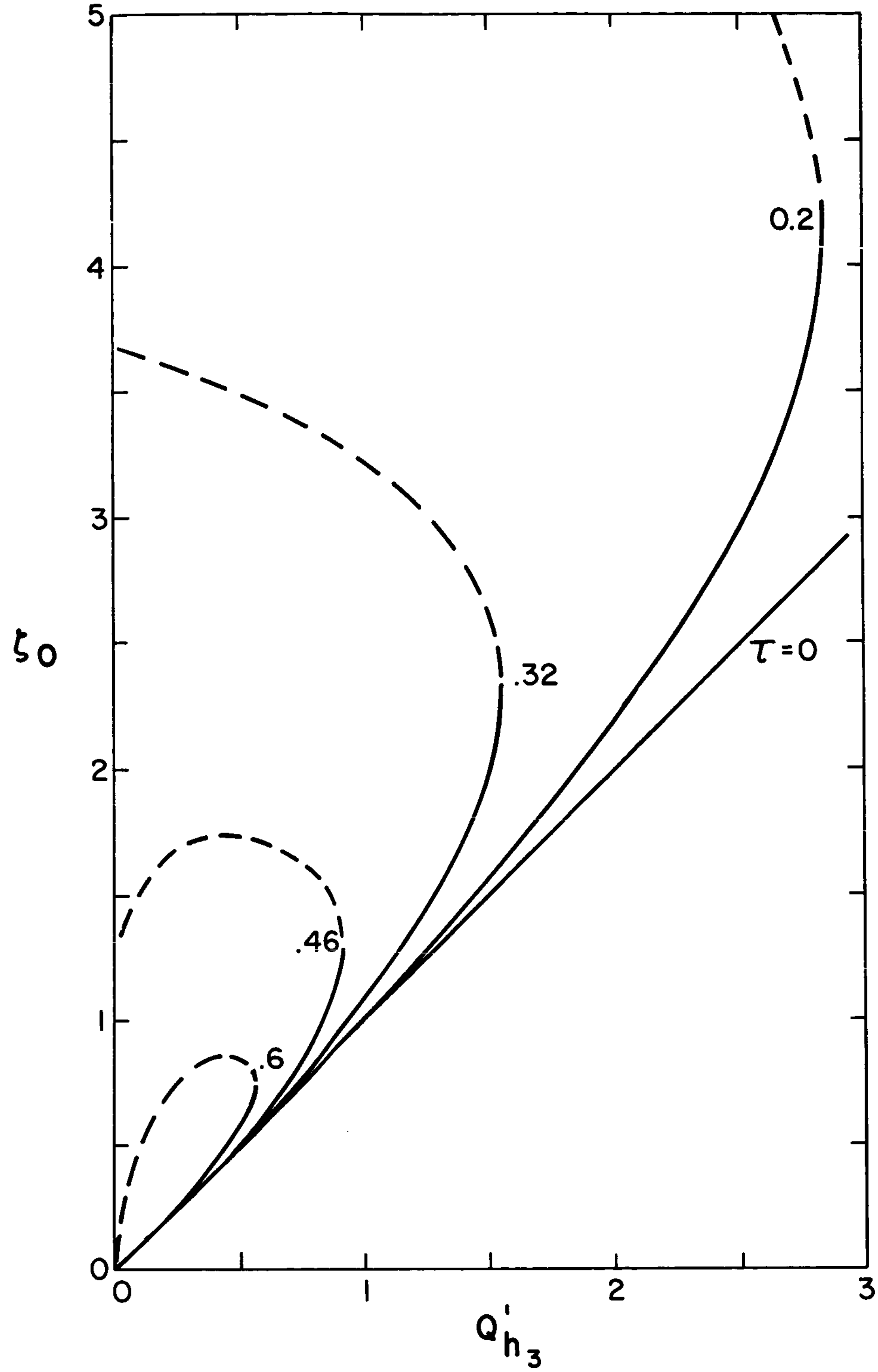

Fig. 10 Dimensionless radius over which the coil is normal versus heat input at the origin. Quench occurs when $\partial \zeta_o / \partial Q'_{h_3} \rightarrow \infty$.

A7448

$$\frac{1}{\zeta^2}\frac{\partial}{\partial\zeta}\left(\zeta^2\frac{\partial\theta_1}{\partial\zeta}\right) - \theta_1 + a\tau^2 = 0, \quad 0<\zeta<\zeta_o \tag{54}$$

2) The region where current is shared $(0 < f < 1)$ is

$$\frac{1}{\zeta^2}\frac{\partial}{\partial\zeta}\left(\zeta^2\frac{\partial\theta_2}{\partial\zeta}\right) - \theta_2(1-a\tau) - a\tau(1-\tau) = 0, \quad \zeta_o<\zeta<\zeta_1 \tag{55}$$

3) The region where all the current is in the superconductor $(f = 0)$ is

$$\frac{1}{\zeta^2}\frac{\partial}{\partial\zeta}\left(\zeta^2\frac{\partial\theta_3}{\partial\zeta}\right) - \theta_3 = 0, \quad \zeta>\zeta_1 \tag{56}$$

The boundary conditions are Eqs. (41), (42), (43), and

$$\zeta^2\left(\frac{d\theta_1}{d\zeta}\right) \rightarrow -Q_{h3} \equiv -\frac{q_{h3}}{4\pi k r_o(T_c - T_b)} \text{ as } \zeta\rightarrow 0 \tag{57}$$

It may be shown that the solution to the three differential equations subject to the stated boundary conditions is:

$$\theta_1 = (\zeta \sinh \zeta_o)^{-1}\left[Q_{h3}\sinh(\zeta_o - \zeta) + \zeta_o(1-a\tau^2)\sinh\zeta\right] + a\tau^2 \tag{58}$$

$$\theta_2 = \zeta^{-1}\left[C_1\sin(\sqrt{a\tau-1}\,\zeta) + C_2\cos(\sqrt{a\tau-1}\,\zeta)\right] - \frac{a\tau(1-\tau)}{1-a\tau}, \text{ for } a\tau>1 \tag{59}$$

$$\theta_2 = \zeta^{-1}\left[C_3\sinh(\sqrt{1-a\tau}\,\zeta) + C_4\cosh(\sqrt{1-a\tau}\,\zeta)\right] - \frac{a\tau(1-\tau)}{(1-a\tau)} \text{ for } a\tau<1 \tag{60}$$

$$\theta_3 = (1-\tau)\,\zeta^{-1}\,\zeta_1\,e^{(\zeta_1-\zeta)} \tag{61}$$

where:

$$C_1 = \frac{\zeta_1 (1 - \tau)}{1 - a\tau} \sin (\sqrt{a \tau - 1}\, \zeta_1) + \frac{(1 - \tau)}{\sqrt{a \tau - 1}} \left[\frac{a\tau}{1 - a\tau} - \zeta_1 \right] \cos (\sqrt{a\tau - 1}\zeta_1) \tag{62}$$

$$C_2 = \frac{\zeta_1(1 - \tau)}{1 - a\tau} \cos (\sqrt{a\tau - 1}\, \zeta_1) - \frac{(1 - \tau)}{\sqrt{a \tau - 1}} \left[\frac{a \tau}{1 - a \tau} - \zeta_1 \right] \sin (\sqrt{a\tau - 1}\zeta_1) \tag{63}$$

$$C_3 = - \frac{\zeta_1(1 - \tau)}{1 - a\tau} \sinh (\sqrt{1 - a\tau}\, \zeta_1) + \frac{(1 - \tau)}{\sqrt{1 - a\tau}} \left[\frac{a \tau}{1 - a \tau} - \zeta_1 \right] \cosh (\sqrt{1 - a\tau}\, \zeta_1) \tag{64}$$

$$C_4 = \frac{\zeta_1 (1 - \tau)}{(1 - a\tau)} \cosh (\sqrt{1 - a\tau}\, \zeta_1) - \frac{(1 - \tau)}{\sqrt{1 - a\tau}} \left[\frac{a\tau}{1 - a \tau} - \zeta_1 \right] \sinh (\sqrt{1 - a\tau}\, \zeta_1) \tag{65}$$

When $a\tau > 1$, the lengths $\zeta_o + \zeta_1$ are determined by

$$\cos \left[\sqrt{a \tau - 1}\, (\zeta_1 - \zeta_o) \right] - \frac{(1 - a\tau)}{\sqrt{a \tau - 1}} \left[\frac{a\tau}{\zeta_1 (1 - a\tau)} - 1 \right] \sin \left[\sqrt{a \tau - 1}\, (\zeta_1 - \zeta_o) \right] = \frac{\zeta_o (1 - a \tau^2)}{\zeta_1 (1 - \tau)} \tag{66}$$

and

$$\sin \left[\sqrt{a\tau - 1}\, (\zeta_1 - \zeta_o) \right] + \frac{1 - a\tau}{\sqrt{a \tau - 1}} \left[\frac{a \tau}{\zeta_1 (1 - a\tau)} - 1 \right] \cos \left[\sqrt{a \tau - 1}\, (\zeta_1 - \zeta_o) \right] = \frac{\sqrt{a \tau - 1}}{\zeta_1 (1 - \tau)} \left[\frac{Q_{h3}}{\sinh \zeta_o} - (1 - a \tau^2) (\cosh \zeta_o - 1) + \frac{1 - a \tau^2}{a \tau - 1} \right] \tag{67}$$

When $\alpha\tau < 1$, the lengths $\zeta_o + \zeta_1$ are determined by

$$\zeta_1 \frac{(1-\tau)}{(1-\alpha\tau)} \cosh \left[\sqrt{1-\alpha\tau}\,(\zeta_1 - \zeta_o)\right] - \frac{(1-\tau)}{\sqrt{1-\alpha\tau}} \left[\frac{\alpha\tau}{1-\alpha\tau} - \zeta_1\right] \sinh \left[\sqrt{1-\alpha\tau}\,(\zeta_1 - \zeta_o)\right] = \zeta_o \frac{(1-\alpha\tau^2)}{(1-\alpha\tau)} \tag{68}$$

and

$$-\zeta_1 \frac{(1-\tau)}{(1-\alpha\tau)} \sinh \left[\sqrt{1-\alpha\tau}\,(\zeta_1 - \zeta_o)\right] + \frac{(1-\tau)}{\sqrt{1-\alpha\tau}} \left[\frac{\alpha\tau}{1-\alpha\tau} - \zeta_1\right] \cosh \left[\sqrt{1-\alpha\tau}\,(\zeta_1 - \zeta_o)\right] = (1-\alpha\tau)^{-1/2}\left[-\frac{Q_{h3}}{\sinh\zeta_o} + (1-\alpha\tau^2)(\coth\zeta_o - 1) + \frac{(1-\alpha\tau^2)}{1-\alpha\tau}\right] \tag{69}$$

From Eqs. (66) and (67), it may now be shown that $\zeta_o \rightarrow \infty$ when $\alpha\tau^3 = 1$. This is analogous to the condition that $V \rightarrow \infty$ when $\alpha\tau^2 = (2-\tau)$ in the one-dimensional situation. This implies that for α, τ such that $\alpha\tau^3 > 1$, quench will occur ($\zeta_o \rightarrow \infty$) when Q_{h3} approaches a take-off value. These conditions for Q_{hT} have not yet been determined, but are being investigated. Other conditions which are needed are those corresponding to the limits of stability and those indicating hysteretic performance.

V. COIL OPERATION AT SUPERCRITICAL PRESSURES*

A series of tests were conducted to gain an insight into any change in the operating characteristics of superconducting coils when cooled with supercritical helium rather than the usual method of cooling with liquid helium at 1 atm. This involved a facility which allowed small coils to be excited and driven to quench while immersed in helium at 4.2°K and pressures from 0 to 45 psig. Since the magnitude of the quench current is dependent on the ability of the heat transfer environment to cool and, in turn, prevent the propagation of a localized normal region (4, 5), these tests provided an indication of the cooling efficiency at elevated pressures relative to that at 0 psig for particular coil constructions.

To create a pressurized coil environment, a vessel was constructed, as shown in Fig. 11, from a short length of brass pipe and caps. The vessel was pressurized through a stainless steel tube which also served as a conduit for the instrumentation leads to the two carbon resistors and for the

*This work was supported in part by the Avco Corporation IRD Program.

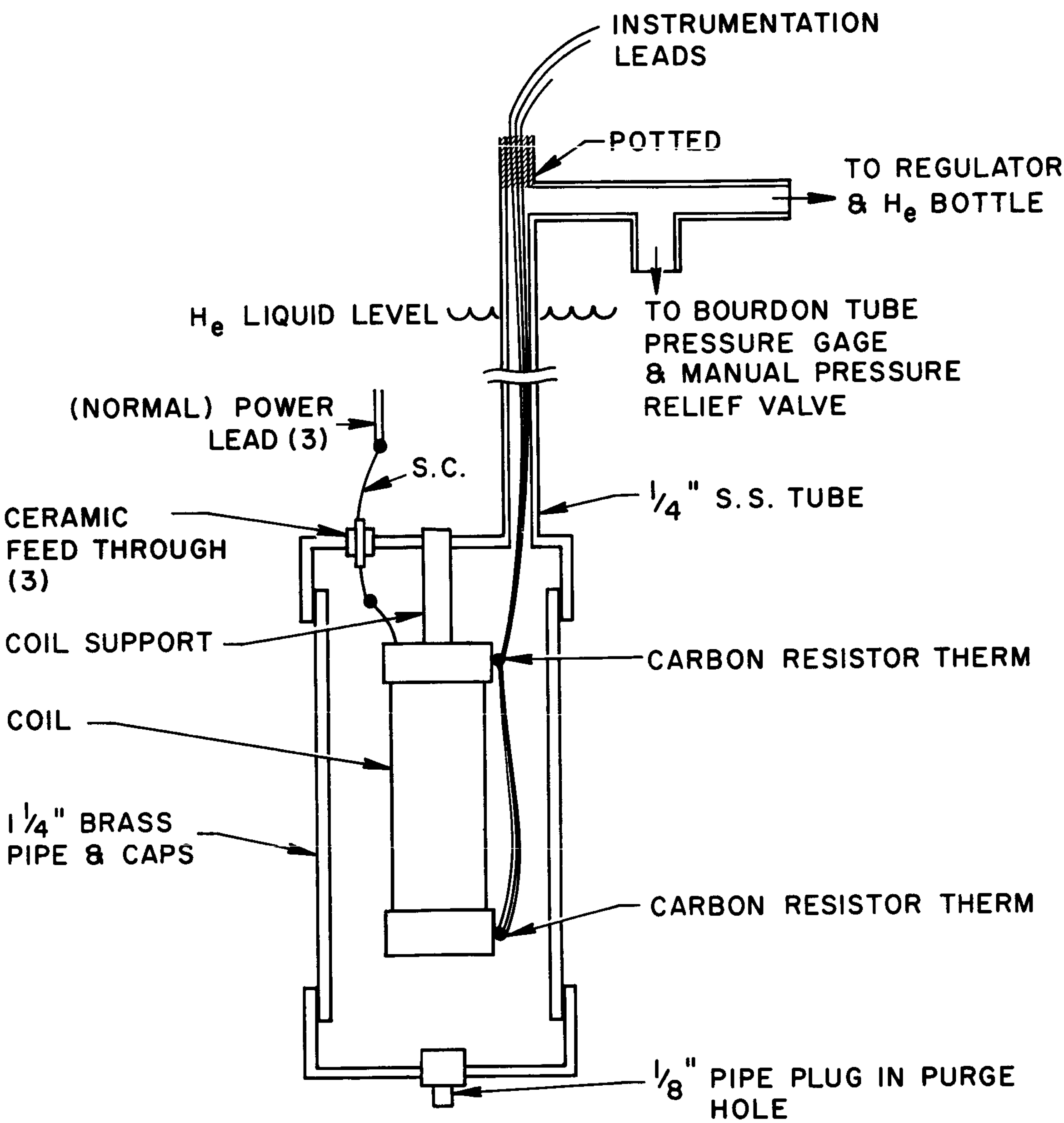

Fig. 11 Experimental Apparatus to Test Coil Operation in a Supercritical Pressure Environment.

A7449

leads to the coil voltage taps. Three superconducting power leads entered the vessel via three ceramic feed-throughs and were solder-joined to the leads on the coil.

The procedure was to mount a coil on the coil support, connect the power and instrumentation leads, assemble the vessel, purge the system with a helium flow from the bottle and out the purge hole, seal the purge hole, leak test the system, pre-cool and immerse the entire assembly in liquid helium. Initially, a great deal of boil-off was observed as the helium from the gas supply was being condensed by the cold walls of the vessel. In each case, it was evident when the liquid level in the vessel rose and covered first the lower temperature probe and then the higher. A short time after the higher probe was covered, boil-off was no longer evident.

Each coil was tested under various initial conditions on pressure. This was controlled via the regulator on the gas bottle and the test began after the upper probe indicated a temperature of $4.2^{\circ}K$, the temperature of the bath. In each case, as coil current was increased, no change in initial pressure or temperature was observed until the coil developed a resistance. After quench, the pressure and temperature rise were dependent on the magnitude of the quench current itself, since this determines the amount of energy dumped into the confined vessel.

Figure 12 indicates typical test results. In each case, quench current decreased somewhat as pressure was increased above 0 psig. For coil B, the change in performance was much more pronounced. This coil exhibited a relatively large voltage prior to quench, hence, the rate of energy deposition to the environment before quench was much larger and the decrease in the ability of the environment to cool the coil at increased pressure is more evident.

The results imply that improved performance may not be expected for coils of the type tested operating at $4.2^{\circ}K$ and pressures between 0 and 40 psig.

CONCLUSION

The previous sections have presented results regarding the behavior of composite conductors. The theoretical model in each section was chosen so as to exemplify the effects of a particular characteristic. It is clear that the zero- and one-dimensional theories are fairly complete, but that the study of three-dimensional effects is still under development.

A test facility is now under construction for the purpose of providing experimental evidence for the confirmation of the predicted behavior. Results will indicate those areas in the developing theory which require extension and refinement as well as those portions which are complete. It is important to note at the outset that the models which are being used, qualitatively exhibit all of the characteristics which are observed in the laboratory.

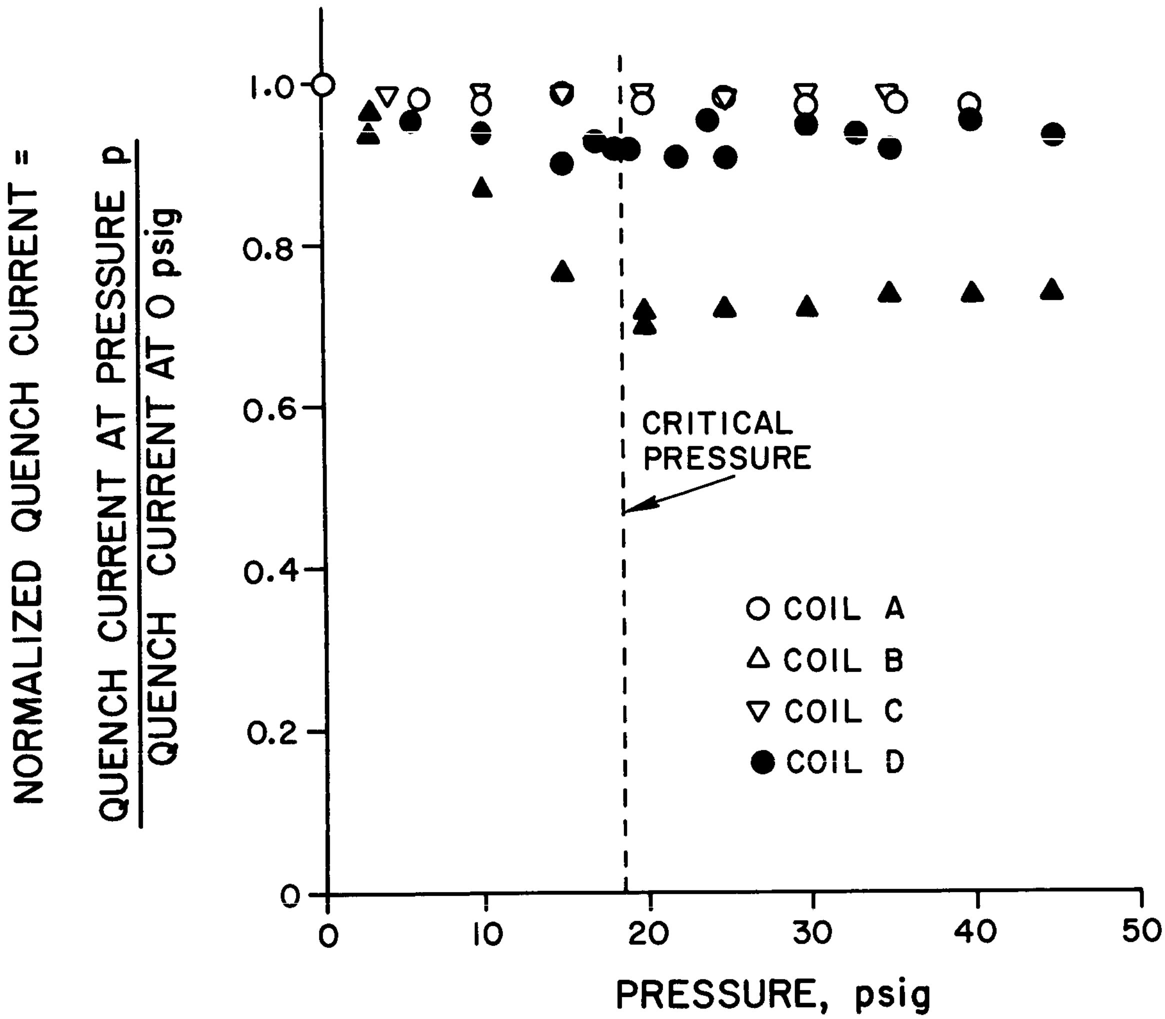

Fig. 12 Normalized Quench Current vs Environmental Pressure.

REFERENCES

1. Hoag, E., et al., "Experimental Investigation of Advanced Superconducting Magnets," Final Report, Contract No. NAS 8-5279, Jan. 1967, Chapter II.

2. Hoag, op. cit., Section II-B.

3. Hoag, op. cit., Section II-C

4. Kantrowitz, A. R. and Stekly, Z. J. J., "A New Principle for the Construction of Stabilized Superconducting Coils," Appl. Phys. Letters, 6, 56, (1965).

5. Cummings, R. D. and Latham, W. N., "An Experimental Measurement of the Dependence of the H-I Curve of Nb-Zr Wire on Temperature," AERL AMP 141, May, 1964.